Parameterization of the Autoconversion Process: Kessler-Type, Sundqvist-Type, and Unification

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Introduction

A key process that must be parameterized in atmospheric models of various scales (from large eddy simulation models to cloud-resolving models to global climate models) is the autoconversion process whereby large cloud droplets collect small ones and become embryonic raindrops. Accurate parameterization and physical understanding of this process is especially important for studies of the second indirect aerosol effect. (Rotstayn 2000; Rotstayn and Liu 2005). All the autoconversion parameterizations that have been developed so far can be generically written as

$$P = P_0 T \tag{1}$$

where P is the autoconversion rate (g cm^{-3} s^{-1}); P_0 represents the autoconversion rate after the onset of the autoconversion process (rate function hereafter), and T represents the threshold function describing the onset of the autoconversion process.

The rate function P_0 has been the primary focus of previous studies (Kessler 1969; Manton and Cotton 1977; Liou and Ou 1989; Baker 1993; Liu and Daum 2004). The threshold function T, however, has received little attention. The only two available expressions are ad hoc in nature (Kessler 1969; Sundqvist 1978; Del Genio et al. 1996). Lack of physics behind these ad hoc threshold functions is a deficiency of existing autoconversion parameterizations.

This work focuses on the threshold function, and is an extension of our two recent studies that derive theoretical expressions for the rate function (Liu and Daum 2004) and the critical radius associated with the Kessler-type parameterizations (Liu et al. 2004). By generalizing Sundqvist-type parameterizations, we show that Kessler-type parameterizations are special cases of the corresponding Sundqvist-type parameterization, unifying the two types of parameterizations. A new threshold function is derived by considering the effect of truncating the cloud droplet size distribution on the autoconversion rate. Combining the new threshold function with the Liu-Daum rate function leads to a new type of autoconversion parameterization that has a firm theoretical basis.

Development of a Kessler-Type Parameterization

Without loss of generality, later Kessler-type parameterizations can be written as

$$P_K = P_0 H\left(r_m - r_c\right) \quad (2)$$

where the subscript "K" means that the corresponding variable is of Kessler-type. The Heaviside step function $H(r_m - r_c)$ is introduced as a threshold function to describe a sudden transition from cloud water to rain water when the driving radius r_m is larger than the critical radius r_c. Previous effort has been mainly devoted to improving specification of the rate function P_0 (see Liu and Daum 2004 for detailed discussions on existing autoconversion functions and their differences). The Liu-Daum rate function is given by

$$P_{LH} = \kappa \beta_6^6 N^{-1} L^3 \quad (3)$$

where κ =1.1 x 10^{10} cm^{-3} s^{-1}, and β_6 is a known function of the relative dispersion of the droplet size distribution.

The critical radius had been artificially tuned until recently when we derived an analytical expression from the kinetic potential theory (Liu et al. 2004; McGraw and Liu 2004)

$$r_c = \left(\frac{3}{4\pi}\right)^{1/3} \frac{\nu^{1/3}\beta_{con}^{1/6}}{\kappa^{1/6}} N^{1/6} L^{-1/3} \quad (4)$$

where ν=3.0 x 10^{-23} (g), and β_{con} = 1.15 x 10^{23} (s^{-1}).

Despite this progress, Kessler-type parameterizations still suffer from the following deficiencies: (1) the discontinuously abrupt transition from cloud water to rainwater described by the Heaviside step function is physically unrealistic; (2) the definition of the driving radius varies for different parameterizations, and (3) the parameterizations implicitly accounts for cloud droplets of all sizes in calculation of the autoconversion rate without considering the effect of truncating the droplet size distribution.

Existing Sundqvist-Type Parameterizations and Generalization

Sundquist (1978) proposed another ad hoc expression for the threshold function

$$T_S = 1 - \exp\left[-\left(\frac{L}{L_c}\right)^2\right] \quad (5)$$

where L_c is the critical liquid water content that is often prescribed in atmospheric models. A slightly different threshold function was introduced by Del Genio et al. (1996)

$$T_S = 1 - \exp\left[-\left(\frac{L}{L_c}\right)^4\right] \quad (6)$$

Equation (5) exhibits a cloud-to-rain transition sharper than Eq. (4), but still smoother than the Heaviside step function. These Sundqvist-type parameterizations can be easily generalized, and the generalized Sundqvist threshold function is given by

$$T_S = 1 - \exp\left[-\left(\frac{\bar{m}}{m_c}\right)^\mu\right] = 1 - \exp\left[-\left(\frac{r_3^3}{r_c^3}\right)^\mu\right] \quad (7)$$

where m = L/N is the mean mass, mc = L/N is the critical mass, and r_3 is the mean-volume radius. The empirical exponent μ > 0 is introduced to unify Kessler-type and Sundqvist-type parameterizations. A new Sundqvist-type parameterization that explicitly accounts for the droplet concentration and relative dispersion in addition to the liquid water content can be obtained by combining the Liu-Daum autoconversion function and the generalized Sundqvist-type threshold function such that

$$P_S = \kappa \beta_6^6 N^{-1} L^3 \left[1 - \exp\left(-\left(\frac{r_3}{r_c}\right)^{3\mu}\right)\right] \quad (8)$$

Unification of Kessler-and Sundqvist-Type Parameterizations

It is clear that the only difference between the Kessler-type parameterization and the generalized Sundqvist-type parameterization lies in their threshold functions. Figure 1 shows that the Kessler-type parameterization becomes a special case of the generalized Sundqvist-type parameterization when m approaches μ.

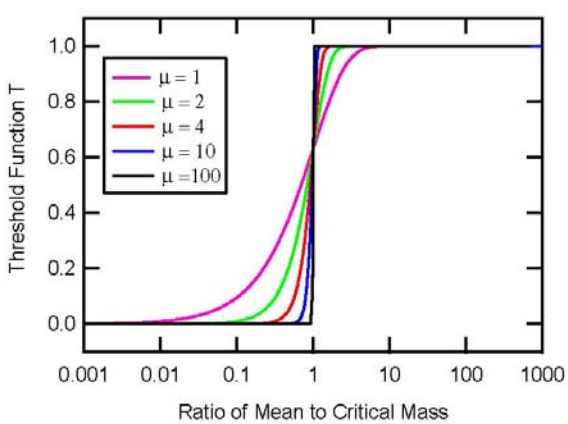

Figure 1. Illustration of the generalized Sundqvist-type threshold function for different values of exponent μ. It is evident that the generalized Sundqvist-type threshold function approaches a Heaviside step function when μ approaches 100.

Size Truncation Effect, Truncation Function, and New Parameterization

Despite the nice features of the generalized Sundqvist-type parameterization, there is no physical basis for the form of the threshold function. It turns out that the threshold function is equivalent to the size truncation function defined below.

All the equations used in Liu and Daum (2004) to derive Eq. (3) hold when truncating the cloud droplet size distribution between rc and rd (rd is the upper truncation radius), except that Eq. (3) becomes

$$P = \kappa \beta_e^6 N_e^{-1} L_e^3 = P_0 \alpha \quad (9a)$$

$$N_e = \int_{m_c}^{m_d} n(m) dm \quad (9b)$$

$$L_e = \int_{m_c}^{m_d} mn(m) dm \quad (9c)$$

$$\beta_e = \left[\frac{\int_{m_c}^{m_d} m^2 n(m)\, dm}{\int_{m_c}^{m_d} n(m)\, dm} \right]^{1/6} \left[\frac{\int_{m_c}^{m_d} mn(m)\, dm}{\int_{m_c}^{m_d} n(m)\, dm} \right]^{-1/3} \quad (9d)$$

where m_c and m_d are the critical mass and the upper truncation mass, respectively, and the quantities with subscript "e" are for the cloud droplets between rc and r_d, whereas their counterparts are for all the droplets (radius from 0 to μ). For a typical exponential droplet size distribution. he size truncation function α defined as

$$\alpha = \alpha_\beta^6 \alpha_N^{-1} \alpha_L^3 \quad (10)$$

To evaluate α (α_β, α_N, and α_L), we employ a typical mass distribution of the Boltzman type (see Liu et al. [1995], Liu and Hallett [1997], and Costa et al. [2000] for justification of using this distribution)

$$n(m) = \frac{N}{\overline{m}} \exp\left(-\frac{m}{\overline{m}} \right) \quad (11)$$

Substitution of Eq. (8) into Eqs. (6b), (6c), and (6d) yields the following expressions:

$$\alpha_N = e^{-x_c} - e^{-x_d} \tag{12a}$$

$$\alpha_L = \left[(x_c + 1) e^{-x_c} - (x_d + 1) e^{-x_d} \right] \tag{12b}$$

$$\alpha_\beta^6 = \frac{1}{2} \frac{\left[(x_c^2 + 2x_c + 2) e^{-x_c} - (x_d^2 + 2x_d + 2) e^{-x_d} \right] \left(e^{-x_c} - e^{-x_d} \right)}{\left[(1 + x_c) e^{-x_c} - (1 + x_d) e^{-x_d} \right]^2} \tag{12c}$$

$$\alpha = \frac{1}{2} \left[(x_c^2 + 2x_c + 2) e^{-x_c} - (x_d^2 + 2x_d + 2) e^{-x_d} \right] \left[(1 + x_c) e^{-x_c} - (1 + x_d) e^{-x_d} \right] \tag{12d}$$

where the normalized upper truncation mass $x_d = \left(\frac{m_d}{\bar{m}} \right)$, and the normalized critical mass $x_c = \left(\frac{m_c}{\bar{m}} \right)$. According to Eq. (9d), for a given x_d, α is a unique function of x_c. When $x_d = \propto$, Eq. (9d) becomes

$$\alpha_\infty = \frac{1}{2} (x_c^2 + 2x_c + 2)(1 + x_c) e^{-2x_c} \tag{12e}$$

Figure 2 shows α as a function of x_c for $x_d = 1$ (dotted), 2 (dashed), 10 (dot-dashed), and ∝ (solid), respectively. It is clear from this figure that α quickly approaches $\alpha_\propto$ after x_d approaches 10 (Note that the dot-dashed curve for $x_d = 10$ is virtually overlaps the solid curve for $\alpha_\propto$). Because the condition that $x_d = 10$, or $r_d = 10^{1/3}\ r_3$, is usually satisfied (e.g., $x_d = 37$ if $r_d = 50$ μm and $r_3 = 15$ μm), it is reasonable to assume that $\alpha = \alpha_\propto$. Furthermore, $\alpha_\propto$ exhibits the threshold behavior expected for the threshold function. The equivalence of the size truncation function and the threshold function is evident from the fact that r_c signals the onset of the autoconversion process. For consistency with the common form of the parameterization (i.e., Eq. [1]), $\alpha_\propto$ is hereafter referred to as the new threshold function and denoted as T_{New}.

Comparisons of Figures 3 with 2, and Eq. (8) with (1) indicate that Eq. (13) is a threshold function. Therefore, an entirely new parameterization can be obtained by combining Eqs. (3), (4), and (11). Figure 3 compares this new parameterization with the corresponding Kessler-type and two typical Sundqvist-type parameterizations.

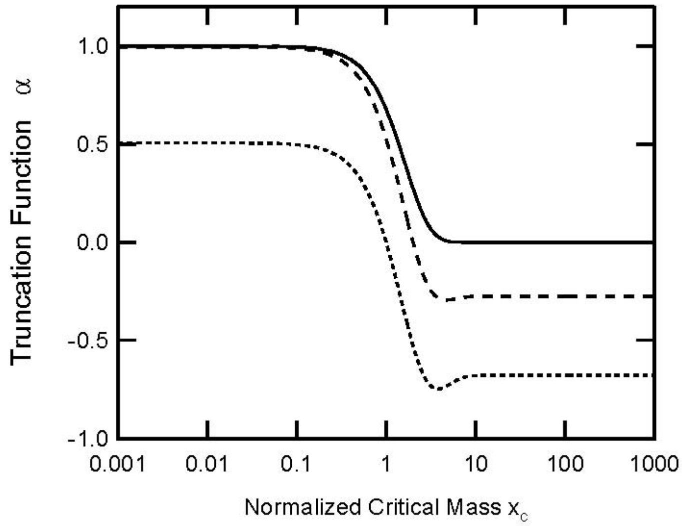

Figure 2. Size truncation function as a function of x_c for x_d = 1 (dotted), 2 (dashed), 10 (dot-dashed), and μ (solid), respectively. Note that the curve for x_d = 10 overlaps with the μ curve.

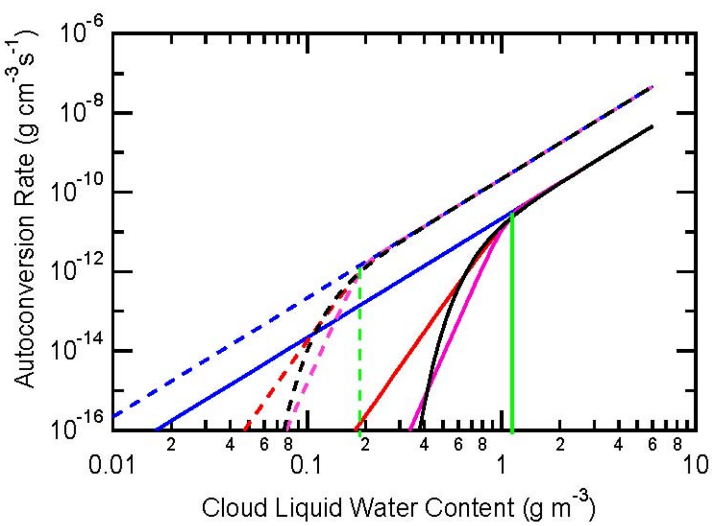

Figure 3. Comparison of the new type of autoconversion parameterization with the existing ones. The dashed and solid curves represent that for N = 50 cm^{-3} and N = 500 cm^{-3}, respectively. The black, green, red and purple colors represent results from the new parameterization, the Kessler-type parameterization with r_3 as the driving radius, and the Sundqvist-type parameterizations with μ = 2, 4, respectively. The blue color simply represents the Liu-Daum autoconversion function, or without the threshold effect. The L_c values (L_c = 0.19 gm^{-3} for N = 50 cm^{-3}; L_c = 1.04 gm^{-3} for N = 500 cm^{-3} are determined by setting x_c = 1 in Equation (16) such that $L_c = 9.8 \times 10^{-9} N^{3/4}$.

Concluding Remarks

It is argued that the autoconversion rate P can be generally expressed as a product of two distinct parts: the autoconversion function P_0 and the threshold function T, and that existing parameterizations can be classified into either Kessler-type or Sundqvist-type according to their threshold functions. Existing Sundqvist-type parameterizations are first generalized by introducing an empirical exponent μ, and then extended to explicitly account for the effect of the cloud droplet concentration on the autoconversion rate. The generalized Sundqvist-type parameterization includes the corresponding Kessler-type parameterization as a limiting case of $\mu \rightarrow \propto$, unifying the two traditionally different types of the autoconversion rate.

A function defined as the size truncation function is introduced to quantify the effect of size truncation of the cloud droplet size distribution on the autoconversion rate. It is shown that the threshold behavior associated with the autoconversion process can be represented by the size truncation function, providing a unified physical explanation for the threshold function. A new type of parameterization is obtained by coupling the new threshold function with our recently derived expressions for the rate function and critical radius. The new parameterization further reveals the approximations and eliminates many deficiencies of the existing Kessler-type and Sundqvist-type parameterizations. For example, in contrast to Kessler-type parameterizations, this new parameterization does not require specification of the empirical driving radius and critical radius. Furthermore, it is shown that Sundqvist-type parameterizations with smooth cloud-to-rain transitions describe the autoconversion rate better than Kessler-type parameterizations with discontinuously sharp transitions. The fact that the autoconversion rate is a product of the rate function and the threshold function also raises questions as to those parameterizations based on fitting numerical results from detailed microphysical simulations with a simple function.

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