Accurate expressions for the dew point and frost point derived from the Rankine-Kirchhoff approximations

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ABSTRACT

Accurate, explicit, and analytic expressions are derived for the dew point and frost point as functions of temperature and relative humidity. These are derived theoretically in terms of physical constants using the Rankine-Kirchhoff approximations, which assume an ideal gas, fixed heat capacities, and zero specific volume of condensates. Compared to modern laboratory measurements, the expressions are accurate to within a few hundredths of a degree over the full range of Earth-relevant temperatures, from 180 to 273 K for the frost point and 230 to 330 K for the dew point.

John Dalton first introduced a method for reliably measuring the dew point, accomplished by determining the highest temperature of water in a glass container that condenses water on its exterior (Dalton 1802b,a). This temperature was originally referred to as the "vapour point" or "point of condensation", but the term "dew point" came into use shortly thereafter (Howard 1818; Dalton 1824). In the years since, many different analytic expressions have been proposed for the dew point in terms of temperature and relative humidity (e.g., Mitchell 1967; Wanielista et al. 1997; Sargent 1980; Urbank et al. 2001; Lawrence 2005), but only one has been derived theoretically and that one, as we will see, has errors that can exceed a degree Kelvin. This short note derives, from first principles, analytic expressions for the dew point and frost point that are accurate to within hundredths of a degree.

The dew point T_d is the temperature to which air must be cooled at constant pressure to reach saturation with respect to liquid water. At the dew point, the vapor pressure p_v equals the saturation vapor pressure with respect to liquid $p_v^{*,l}(T_d)$; here, the asterisk denotes saturation and the superscript l reminds us that this is with respect to a planar surface of liquid. In the absence of phase changes, the partial pressure of water vapor does not change as the air is cooled isobarically, so we can write p_v as $\mathrm{RH}_l p_v^{*,l}(T)$, where T is the original

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temperature of the air and RH_l is the original relative humidity of the air with respect to liquid water. Therefore, $p_v^*(T_d) = \mathrm{RH}_l p_v^{*,l}(T)$, which can be written as (e.g., Bosen 1958)

$$\mathrm{RH}_l = \frac{p_v^{*,l}(T_d)}{p_v^{*,l}(T)}\,. \tag{1}$$

Given an expression for the function $p_v^{*,l}$, equation (1) gives RH_l as an explicit function of temperature T and dew point T_d. Likewise, given the frost point T_f, we can calculate the relative humidity with respect to solid water RH_s as

$$\mathrm{RH}_s = \frac{p_v^{*,s}(T_f)}{p_v^{*,s}(T)}\,, \tag{2}$$

where $p_v^{*,s}$ is the saturation vapor pressure with respect to a planar surface of solid water, i.e., ice.

Given T and RH_l (or RH_s), the most accurate value of the dew (or frost) point is obtained by using an empirical expression for $p_v^{*,l}$ (or $p_v^{*,s}$) that has been obtained by fitting to laboratory measurements such as equation (10) of Murphy and Koop (2005) (or equation (4) of Wagner et al. 2011) and then solving for T_d (or T_f) using a numerical root solver. Our goal here, however, is to derive expressions for T_d and T_f that are not only accurate, but also explicit and analytic in terms of temperature and relative humidity. Therefore, we must invert $p_v^{*,l}$. For this purpose, it is common to use the empirical Magnus expression for $p_v^{*,l}$ (August 1828; Magnus 1844), which can be inverted analytically (Mitchell 1967; Lawrence 2005). But we have here one other objective, which is to derive an expression for the dew point that is not only accurate, explicit, and analytic, but also derived theoretically.

To this end, we will use the approximations of Rankine (1866) and Kirchhoff (1858), which assume an ideal gas, constant heat capacities, and zero specific volume of condensates. Following Romps (2021), we may refer to this trio of assumptions as the Rankine-Kirchhoff (RK) approximations, which have been used to derive accurate, explicit, and analytic expressions for the equivalent potential temperature (Romps and Kuang 2010), the quantity conserved by an adiabatically lifted parcel (Romps 2015), and the lifting condensation level (Romps 2017). Here, we aim to add to that list the dew point and frost point.

Using the Rankine-Kirchhoff approximations, we can arrive at the Rankine-Kirchhoff equations for the saturation vapor pressures (Rankine 1866; Kirchhoff 1858)

$$p_v^{*,l} = p_{\mathrm{trip}}\left(\frac{T}{T_{\mathrm{trip}}}\right)^{\frac{c_{pv}-c_{vl}}{R_v}} \exp\left[\frac{E_{0v} - (c_{vv}-c_{vl})T_{\mathrm{trip}}}{R_v}\left(\frac{1}{T_{\mathrm{trip}}} - \frac{1}{T}\right)\right] \tag{3}$$

$$p_v^{*,s} = p_{\mathrm{trip}}\left(\frac{T}{T_{\mathrm{trip}}}\right)^{\frac{c_{pv}-c_{vs}}{R_v}} \exp\left[\frac{E_{0v} + E_{0s} - (c_{vv}-c_{vs})T_{\mathrm{trip}}}{R_v}\left(\frac{1}{T_{\mathrm{trip}}} - \frac{1}{T}\right)\right]\,, \tag{4}$$

where R_v is the specific gas constant for water vapor; c_{vv}, c_{vl}, c_{vs} are the specific heat capacities at constant volume of water vapor, liquid water, and solid water, respectively; $c_{pv} = c_{vv} + R_v$ is the specific heat capacity of water vapor at constant pressure; p_{trip} and T_{trip} are the triple-point vapor pressure and temperature, respectively; E_{0v} is the difference in specific internal energy between water vapor and liquid water at the triple point; and E_{0s} is the difference in specific internal energy between liquid water and solid water at the triple point. Following Romps (2017), a set of optimized values for these constants are $c_{vv} = 1418$, J kg^{-1} K^{-1}, $p_{\text{trip}} = 611.65$ Pa, $T_{\text{trip}} = 273.16$ K, $E_{0v} = 2.3740$ MJ kg^{-1}, $E_{0s} = 0.3337$ MJ kg^{-1}, $R_v = 461$ J kg^{-1} K^{-1}, $c_{vl} = 4119$ J kg^{-1} K^{-1}, and $c_{vs} = 1861$ J kg^{-1} K^{-1}.

Using equation (3), equation (1) can be inverted to give an explicit and analytic expression for the dew point T_d,

$$T_d = c\left[W_{-1}\left(\mathrm{RH}_l^{\frac{R_v}{c_{vl}-c_{pv}}} c e^c\right)\right]^{-1} T \tag{5}$$

$$c \equiv \frac{E_{0v} - (c_{vv} - c_{vl})T_{\text{trip}}}{(c_{pv} - c_{vl})T}\,, \tag{6}$$

where W_{-1} is the lower branch of the Lambert W function. Likewise, using equation (4), equation (2) can be inverted to give the frost point T_f,

$$T_f = c\left[W_0\left(\mathrm{RH}_s^{\frac{R_v}{c_{vs}-c_{pv}}} c e^c\right)\right]^{-1} T \tag{7}$$

$$c \equiv \frac{E_{0v} + E_{0s} - (c_{vv} - c_{vs})T_{\text{trip}}}{(c_{pv} - c_{vs})T}\,, \tag{8}$$

where W_0 is the principal branch of the Lambert W function. The reason the dew point and frost point use different branches is that $c_{vl} - c_{pv}$ and $c_{vs} - c_{pv}$ have different signs for the choice of heat capacities used here. If different heat capacities were used for which $c_{vs} > c_{pv}$, then the lower branch would be used for the frost point instead.

In the case of the frost point, the value of e^c can be very large and can cause the argument of W_0 to exceed the largest number that can be represented using the IEEE double-precision floating-point format, which is about 1.8×10^{308}. If the logarithm of the argument exceeds $709 \approx \log(10^{308})$, then we can use an approximation for W_0,

$$W_0(x) \approx L_1 - L_2 + \frac{L_2}{L_1} + \frac{L_2(-2+L_2)}{2L_1^2} + \frac{L_2(6-9L_2+2L_2^2)}{6L_1^3} + \frac{L_2(-12+36L_2-22L_2^2+3L_2^3)}{12L_1^4}\,,$$

where $L_1 = \log(x)$ and $L_2 = \log(L_1)$ (Corless et al. 1996). When L_1 exceeds 709, this approximation replicates W_0 with a relative error that is smaller than double-precision rounding error.

Equations (7–8) give the frost point in terms of RH_s, but relative humidity is most commonly stated with respect to liquid water in the field of atmospheric science. To calculate T_f from RH_l, we can note that $p_v = \mathrm{RH}_l p_v^{*,l} = \mathrm{RH}_s p_v^{*,s}$, and so

$$\frac{\mathrm{RH}_s}{\mathrm{RH}_l} = \frac{p_v^{*,l}}{p_v^{*,s}} = \left(\frac{T}{T_{\text{trip}}}\right)^{\frac{c_{vs}-c_{vl}}{R_v}} \exp\left[\frac{E_{0s} + (c_{vs} - c_{vl})T_{\text{trip}}}{R_v}\left(\frac{1}{T} - \frac{1}{T_{\text{trip}}}\right)\right]. \tag{9}$$

This relation allows us to convert from RH_l to RH_s or vice versa, enabling the calculation of both T_d and T_f no matter which relative humidity is given.

Before we calculate the errors in expressions for the dew and frost points, let us first estimate the uncertainties from modern laboratory measurements. We can estimate the empirical uncertainty in the dew point by taking the reported relative uncertainty in laboratory measurements of the vapor pressure $\Delta \log p_v^{*,l}$ and dividing by $L_e/(R_v T^2)$, where L_e is the latent heat of evaporation. (This estimation is used here as an alternative to propagating errors in the standard way because it is not known to what degree the reported uncertainties in the vapor pressures at T and T_d are correlated.) Similarly, we can estimate the uncertainty in the frost point using the $\Delta \log p_v^{*,s}$ reported for laboratory measurements and dividing by $(L_e + L_m)/(R_v T^2)$, where L_m is the latent heat of melting. This procedure gives an uncertainty of ∼0.003 K at temperatures around 300 K (using $\Delta \log p_v^{*,l} = 2 \times 10^{-4}$ from Figure 6.4 of Wagner and Pruß 2002) and that grows to ∼0.02 K at temperatures around 252 K (using $\Delta \log p_v^{*,l} = 2 \times 10^{-3}$ from Table 2 of Beltramino et al. 2020) and ∼0.03 K at temperatures around 180 K (using $\Delta \log p_v^{*,s} = 5 \times 10^{-3}$ from Figure 5 of Wagner et al. 2011). Note that 252 K was the lowest temperature measured by Beltramino et al. (2020) for supercooled water, so this is a conservative lower bound on the dew-point uncertainty at lower temperatures. In summary, we see that uncertainties in laboratory measurements lead to uncertainties in the dew and frost points as high as a few hundredths of a degree.

As mentioned at the beginning, there is one expression for the dew point already in the literature that is analytic, explicit in terms of temperature and relative humidity, and derivable from theory. That equation is derived by adding to the RK approximations the further assumption that all phases of water have the same specific heat capacity, i.e., $c_{pv} = c_{vl} = c_{vs}$. In that case, equation (3) reduces to (e.g., Callendar 1911; Murphy and Koop 2005)

$$p_v^{*,l} = p_{\text{trip}} \exp\left[\frac{L_0}{R_v}\left(\frac{1}{T_{\text{trip}}} - \frac{1}{T}\right)\right] \qquad (c_{pv} = c_{vl} \text{ approximation})\,,$$

where the latent heat of evaporation is approximated as $L_{e0} = E_{0v} + R_v T_{\text{trip}}$. Using $\text{RH}_l = p_v^{*,l}(T_d)/p_v^{*,l}(T)$ and solving for T_d, we get (Lawrence 2005)

$$T_d = \left(\frac{1}{T} - \frac{R_v}{L_0}\log \text{RH}_l\right)^{-1} \qquad (c_{pv} = c_{vl} \text{ approximation})\,.$$

A similar expression may be obtained for T_f, replacing RH_l with RH_s and replacing the latent heat of evaporation with the approximated sum of the latent heats of evaporation and sublimation, $L_{e0} + L_{m0} = E_{0v} + E_{0s} + R_v$. When calculating these expressions, the values used for E_{0v}, E_{0s}, and R_v are the same as those given above.

The errors in these equal-heat-capacity dew-point and frost-point expressions can be calculated by subtracting from them the dew points and frost points obtained (by use of a root solver) from laboratory measurements of the saturation vapor pressure. For this purpose, we will use equation (10) of Murphy and Koop (2005) for $p_v^{*,l}$, which fits the empirical data as well or better (see Figure 2 of Beltramino et al. 2020) than the IAPWS G12-15 formulation (IAPWS 2015), and we will use equation (4) of Wagner et al. (2011) for $p_v^{*,s}$, which was adopted as IAPWS R14-08 (IAPWS 2011). The dashed curves in Figure

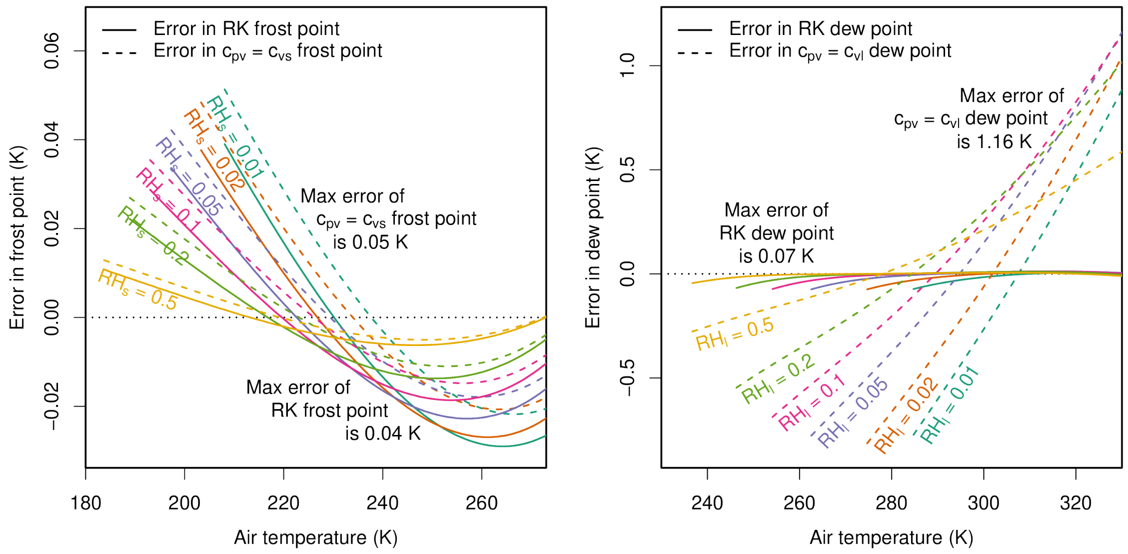

FIG. 1. Errors in the (left) frost point and (right) dew point as obtained from the Rankine-Kirchhoff approximations (solid) or the approximation of equal heat capacities (dashed). Note that the ordinates differ between the left and right panels. Errors are calculated relative to laboratory measurements of vapor pressure summarized by the expressions of Murphy and Koop (2005) for liquid and Wagner et al. (2011) for ice. See the text for a comparison of these errors to the uncertainties from laboratory measurements.

1 show the errors in the equal-heat-capacity frost point (left panel) and dew point (right panel) plotted as functions of air temperature for a variety of relative humidities. Note that the two panels use different ordinates. For the frost point, the curves are plotted for values of T such that both T and T_d lie between 180 and 273 K. For the dew point, the curves are plotted for values of T such that both T and T_d lie between 230 and 330 K.

The error in the equal-heat-capacity expression for the frost point is less than or equal to 0.05 K, which is the same order of magnitude as the uncertainty from laboratory measurements. For the dew point, however, the assumption of $c_{pv} = c_{vl}$ generates errors that far exceed the uncertainties from empirical data. For low relative humidities (less than or equal to about 0.2) or for high air temperatures (exceeding about 310 K, 37 °C, or 100 °F), the dew-point error can approach and exceed 0.5-1.0 °C or 1-2 °F.

Repeating this calculation for the Rankine-Kirchhoff expressions in equations (5) and (7) gives the solid curves in Figure 1. We see that the errors in the RK dew point and RK frost point are less than 0.04 K and 0.07 K, respectively. These errors are the same order of magnitude as the uncertainties from the laboratory measurements. With maximum errors measured in hundredths of a Kelvin, the Rankine-Kirchhoff expressions for the dew point and frost point are sufficiently accurate for most, if not all, atmospheric applications.

To give a graphical summary of the Rankine-Kirchhoff dew-point and frost-point depressions, Figure 2 plots $\min[T - T_d(T), T - T_f(T)]$ (i.e., the smaller of either the dew-point depression or the frost-point depression) as a function of temperature and relative humidity (with respect to the saturation vapor pressure over liquid for $T \geq 273.16$ K and over solid for $T < 273.16$ K). The kink in the curves at a temperature of 273.16 K marks the

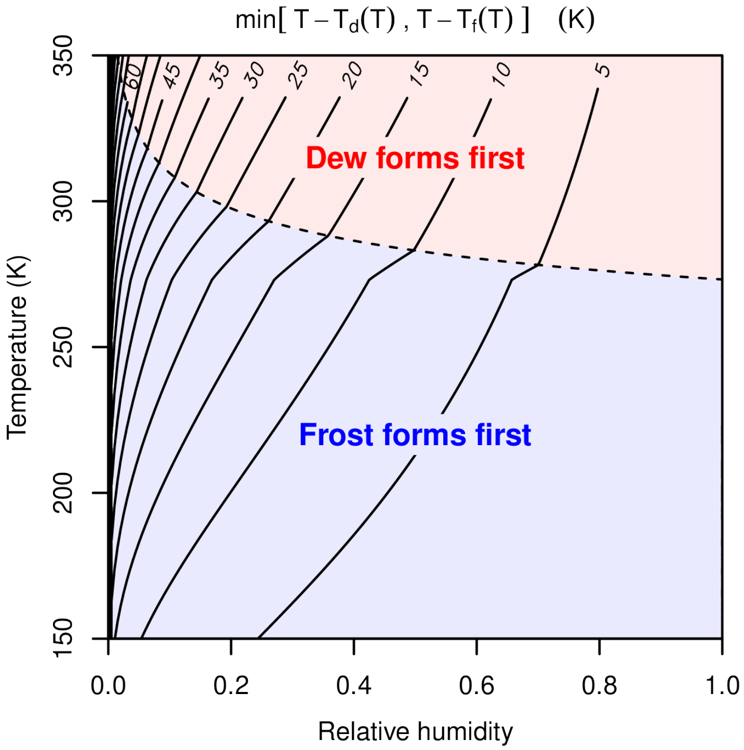

FIG. 2. Minimum of the dew-point depression and frost-point depression plotted as a function of temperature and relative humidity (with respect to the saturation vapor pressure over liquid for $T \geq 273.16$ K and over solid for $T < 273.16$ K). The dashed line separates the regions where, as the air is cooled, (red) dew forms first and (blue) frost forms first.

transition from relative humidity with respect to solid (below the kink) and with respect to liquid (above the kink). A second locus of kinks, marked by the dashed line, occurs where the dew-point depression and frost-point depression are equal. Above the dashed line, dew forms before frost as the air is cooled, and vice versa below the dashed line, assuming readily available surfaces for both condensation and deposition. Code to evaluate equations (5) and (7) are available on the author's web site in R, Python, Fortran, and MATLAB.

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