

# **Light Scattering and Absorption by Spherical Particles in an Absorbing Medium**

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## **Introduction**

Light scattering and absorption by spherical particles in an absorbing medium occur in the earth-atmosphere system. Some examples are light scattering by cloud particles surrounded by the water vapor in the atmosphere, by biological particles in the ocean, and by air bubbles in the ocean and sea ice. It is important for many applications to understand the effects of the absorbing medium on the scattering and absorption of light by the particles.

The light scattering by a spherical particle in a nonabsorbing medium is well understood based on Mie theory (Mie 1980). When the host medium is absorbing, however, Mie equations need to be modified to describe the light scattering by particles. Several studies (Mundy et al. 1974; Chylek 1977; Bohren and Gilra 1979) were carried out to develop the theoretical treatment for light scattering by particles embedded in absorbing media using the far-field approximation. But because of the absorption in the host medium, the extinction, scattering, and absorption efficiencies based on the far-field approximation depend on the radius of the conceptual integrating sphere, which do not represent the actual extinction, scattering, and absorption efficiencies of the particle.

In this work, the light scattering based on Mie theory is formulated for a spherical particle in an absorbing medium. We derive the analytic expressions for single-scattering properties of the particle including the absorption, scattering, and extinction efficiencies, by using the electromagnetic fields on the surface of the particle. This approach avoids the difficulty to derive the extinction from the optical theorem using the far field.

## **Formal Scattering Solution for a Sphere in an Absorbing Medium**

The formal scattering solution for a sphere in a nonabsorbing medium based on Mie theory can be expressed in a form of equations that explicitly exhibit the refractive index of the medium as well as that of the sphere. These equations are formally identical to those obtained for the case of an absorbing medium, where both the refractive indices can be complex. In this section, the formal scattering solution is briefly reviewed by using the notation of Bohren and Huffman (1983).

Consider the scattering of a linearly polarized plane wave by a sphere with a radius,  $a$ , imbedded in an absorbing medium. In this study, we select the origin of a Cartesian coordinate system at the center of the sphere, with the positive  $Z$  axis along the direction of propagation of the incident wave. The incident electric vector is polarized in the direction of the  $x$  axis. In spherical coordinates, if the amplitude of the incident wave at the origin is  $E_0$ , the incident (i), internal (t), and scattered (s) fields can be expressed in spherical harmonics in the form

$$\vec{E}_i = \sum_{n=1}^{\infty} E_n (\vec{M}_{o1n}^{(1)} - i\vec{N}_{e1n}^{(1)}), \quad (1a)$$

$$\vec{H}_i = \frac{-k}{\omega\mu} \sum_{n=1}^{\infty} E_n (\vec{M}_{e1n}^{(1)} + i\vec{N}_{o1n}^{(1)}), \quad (1b)$$

$$\vec{E}_t = \sum_{n=1}^{\infty} E_n (c_n \vec{M}_{o1n}^{(1)} - id_n \vec{N}_{e1n}^{(1)}), \quad (2a)$$

$$\vec{H}_t = \frac{-k_t}{\omega\mu_t} \sum_{n=1}^{\infty} E_n (d_n \vec{M}_{e1n}^{(1)} + ic_n \vec{N}_{o1n}^{(1)}), \quad (2b)$$

$$\vec{E}_s = \sum_{n=1}^{\infty} E_n (ia_n \vec{N}_{e1n}^{(3)} - b_n \vec{M}_{o1n}^{(3)}), \quad (3a)$$

$$\vec{H}_s = \frac{k}{\omega\mu} \sum_{n=1}^{\infty} E_n (ib_n \vec{N}_{o1n}^{(3)} + a_n \vec{M}_{e1n}^{(3)}), \quad (3b)$$

where  $i = \sqrt{-1}$ ,  $E_n = i^n(2n+1)/(n(n+1))E_0$ ;  $\omega$  is the angular frequency;  $k=2\pi m/\lambda_0$  and  $k_t=2\pi m_t/\lambda_0$  with  $\lambda_0$  the wavelength in vacuum, and  $m$  and  $m_t$  the refractive indices of the host medium and the scatterer, respectively;  $\mu$  and  $\mu_t$  are the permeabilities of the host medium and the scatterer, respectively.

The vector spherical harmonics in Eqs. (1)-(3) are given by

$$\vec{M}_{o1n} = \cos\phi\pi_n(\cos\theta)z_n(\rho)\vec{e}_\theta - \sin\phi\tau_n(\cos\theta)z_n(\rho)\vec{e}_\phi, \quad (4a)$$

$$\vec{M}_{e1n} = -\sin\phi\pi_n(\cos\theta)z_n(\rho)\vec{e}_\theta - \cos\phi\tau_n(\cos\theta)z_n(\rho)\vec{e}_\phi, \quad (4b)$$

$$\begin{aligned} \vec{N}_{o1n} = & \sin\phi n(n+1)\sin\theta\pi_n(\cos\theta)z_n(\rho)/\rho\vec{e}_r \\ & + \sin\phi\tau_n(\cos\theta)[\rho z_n(\rho)]'/\rho\vec{e}_\theta + \cos\phi\pi_n(\cos\theta)[\rho z_n(\rho)]'/\rho\vec{e}_\phi, \end{aligned} \quad (4c)$$

$$\begin{aligned} \vec{N}_{eln} = & \cos \phi n(n+1) \sin \theta \pi_n(\cos \theta) z_n(\rho) / \rho \vec{e}_r \\ & + \cos \phi \tau_n(\cos \theta) [\rho z_n(\rho)]' / \rho \vec{e}_\theta - \sin \phi \pi_n(\cos \theta) [\rho z_n(\rho)]' / \rho \vec{e}_\phi, \end{aligned} \quad (4d)$$

where  $\pi_n = P_n^1 / \sin \theta$  and  $\tau_n = dP_n^1 / d\theta$  with  $P_n^1$  the associated Legendre functions of the first kind of degree  $n$  and order one;  $\vec{e}_\theta$ ,  $\vec{e}_\phi$ , and  $\vec{e}_r$  are unit vectors in spherical coordinates;  $\rho = kr$  for the incident and scattered fields and  $\rho = k_1 r$  for the internal fields.

Superscripts appended to  $\vec{M}$  and  $\vec{N}$  in Eqs. (1)-(3) denote the kind of spherical Bessel function  $z_n$ :

(1) denotes  $j_n(\rho)$ , which is defined as  $\sqrt{\frac{\pi}{2\rho}} J_{n+1/2}(\rho)$  where  $J_{n+1/2}$  is the Bessel function of first kind;

(3) denotes  $h_n^{(1)}$ , which is defined as  $j_n(\rho) + iy_n(\rho)$  where  $y_n(\rho) = \sqrt{\frac{\pi}{2\rho}} Y_{n+1/2}(\rho)$  with  $Y_{n+1/2}$  the Bessel function of the second kind.

Using the boundary conditions at the particle-medium interface, the coefficients  $a_n$ ,  $b_n$ ,  $c_n$ , and  $d_n$  in Eqs. (2) and (3) can be derived in the form

$$a_n = \frac{m_t \psi'_n(\alpha) \psi_n(\beta) - m \psi_n(\alpha) \psi'_n(\beta)}{m_t \xi'_n(\alpha) \psi_n(\beta) - m \xi_n(\alpha) \psi'_n(\beta)}, \quad (5a)$$

$$b_n = \frac{m_t \psi_n(\alpha) \psi'_n(\beta) - m \psi'_n(\alpha) \psi_n(\beta)}{m_t \xi_n(\alpha) \psi'_n(\beta) - m \xi'_n(\alpha) \psi_n(\beta)}, \quad (5b)$$

$$c_n = \frac{m_t \xi_n(\alpha) \psi'_n(\alpha) - m_t \xi'_n(\alpha) \psi_n(\alpha)}{m_t \xi_n(\alpha) \psi'_n(\beta) - m \xi'_n(\alpha) \psi_n(\beta)}, \quad (5c)$$

$$d_n = \frac{m_t \xi'_n(\alpha) \psi_n(\alpha) - m_t \xi_n(\alpha) \psi'_n(\alpha)}{m_t \xi'_n(\alpha) \psi_n(\beta) - m \xi_n(\alpha) \psi'_n(\beta)}, \quad (5d)$$

where  $\alpha = ka$ ,  $\beta = k_1 a$ , and the Riccati-Bessel functions  $\psi_n(\rho) = \rho j_n(\rho)$  and  $\xi_n(\rho) = \rho h_n^{(1)}(\rho)$ .

## Absorption, Scattering, and Extinction Efficiencies

For a nonabsorbing host medium, the far-field approximation for the electromagnetic field is usually used to calculate the particle scattering and extinction cross sections. The far-field approach has also been used by Mundy et al. (1974), Chylek (1977), and Bohren and Gilra (1979) to study the scattering and absorption by a spherical particle in an absorbing host medium. When the medium is absorbing, however, the scattering and extinction based on the far-field approximation depend on the radius of the conceptual integrating sphere, which is concentric with the scatterer but with  $r \gg a$ . In this study, we

derive the absorption, scattering, and extinction efficiencies of a sphere embedded in an absorbing medium in such a way that these efficiencies depend only on the size of the particle and optical properties of the particle and surrounding medium. This goal is achieved by using the near field at the surface of the scatterer. The analytic expressions for the absorption, scattering, and extinction efficiencies will be presented.

The flow of energy and the direction of the electromagnetic wave propagation are represented by the Poynting vector

$$\vec{S} = 1/2 \operatorname{Re}(\vec{E} \times \vec{H}^*), \quad (6)$$

where  $\operatorname{Re}()$  represents the real part of the argument; the asterisk denotes the complex conjugate value;  $\vec{E}$  and  $\vec{H}$  are the total electric and magnetic vectors, respectively; and  $|\vec{S}|$  is in the units of flux density.

Using the Poynting vector, the rate of energy absorbed by the spherical particle,  $W_a$ , can be expressed as

$$W_a = -\frac{1}{2} \operatorname{Re} \iint (\vec{E} \times \vec{H}^*) \cdot \vec{n} ds, \quad (7)$$

where the integral is taken over the surface of the scatterer, and  $\vec{n}$  is an outward unit vector normal to the surface of the scatterer. Following the boundary conditions of the electromagnetic fields across the particle surface, we can use either the external fields (the scattered plus incident wave) or the internal fields to evaluate  $W_a$  from Eq. (7). Here we employ the latter because of the simplicity. Therefore,  $W_a$  can be written in the form

$$\begin{aligned} W_a &= -\frac{1}{2} \operatorname{Re} \iint (\vec{E}_t \times \vec{H}_t^*) \cdot \vec{n} ds \\ &= \frac{1}{2} \operatorname{Re} \int_0^{2\pi} \int_0^\pi (E_{t\phi} H_{t\theta}^* - E_{t\theta} H_{t\phi}^*) a^2 \sin \theta d\theta d\phi, \end{aligned} \quad (8)$$

where the internal electromagnetic vector components are obtained from Eqs. (2) and (4).

We have shown that (Fu and Sun, in preparation).

$$W_a = \frac{\pi |E_0|^2}{\omega \mu_t} \sum_{n=1}^{\infty} (2n+1) \operatorname{Im}(A_n), \quad (9a)$$

where  $\operatorname{Im}()$  represents the imaginary part of the argument, and

$$A_n = \frac{|c_n|^2 \psi_n(\beta) \psi_n^*(\beta) - |d_n|^2 \psi_n'(\beta) \psi_n^*(\beta)}{k_t}. \quad (9b)$$

The rate of energy scattered by the scatterer,  $W_s$ , is given by Bohren and Huffman (1983)

$$W_s = \frac{1}{2} \text{Re} \iint (\vec{E}_s \times \vec{H}_s^*) \cdot \vec{n} ds$$

$$= \frac{1}{2} \text{Re} \int_0^{2\pi} \int_0^\pi (E_{s\theta} H_{s\phi}^* - E_{s\phi} H_{s\theta}^*) a^2 \sin \theta d\theta d\phi \quad (10)$$

We have shown that (Fu and Sun, in preparation)

$$W_s = \frac{\pi |E_0|^2}{\omega \mu} \sum_{n=1}^{\infty} (2n+1) \text{Im}(B_n), \quad (11a)$$

where

$$B_n = \frac{|a_n|^2 \xi_n'(\alpha) \xi_n^*(\alpha) - |b_n|^2 \xi_n(\alpha) \xi_n'^*(\alpha)}{k} \quad (11b)$$

The rate of energy extincted by the spherical particle is  $W_e = W_a + W_s$ , which can be written in the form

$$W_e = \frac{\pi |E_0|^2}{\omega} \sum_{n=1}^{\infty} (2n+1) \text{Im} \left( \frac{A_n}{\mu_t} + \frac{B_n}{\mu} \right) \quad (12)$$

Following Mundy et al. (1974), the rate of energy incident on the scatterer in an absorbing medium is

$$f = \frac{2\pi a^2}{\eta^2} I_0 [1 + (\eta - 1)e^\eta], \quad (13)$$

where  $\eta = 4\pi a m_i / \lambda_0$  and  $I_0 = \frac{m_r}{2c\mu} |E_0|^2$ . Here  $c$  is the speed of light in vacuum; and  $m_r$  and  $m_i$  are the real and imaginary parts of the complex refractive index of the host medium, respectively. It can be shown that  $f = \pi a^2 I_0$  when  $m_i = 0$ . Therefore, the absorption, scattering, and extinction efficiencies are

$$Q_a = W_a / f, \quad (14a)$$

$$Q_s = W_s / f, \quad (14b)$$

$$Q_e = W_e / f. \quad (14c)$$

It should be noted that the results presented in this section are generally valid for a sphere embedded in either an absorbing or a nonabsorbing medium.

## Scattering Phase Function and Asymmetry Factor

The scattering phase function represents the angular distribution of scattered energy at very large distances from the sphere, which can be derived using the far-field approximation. It is convenient to define amplitude functions  $S_1$  and  $S_2$  as in Kerker (1969)

$$S_1 = \sum_{n=1}^{\infty} \frac{2n+1}{n(n+1)} [a_n \pi_n(\cos \theta) + b_n \tau_n(\cos \theta)], \quad (15a)$$

$$S_2 = \sum_{n=1}^{\infty} \frac{2n+1}{n(n+1)} [a_n \tau_n(\cos \theta) + b_n \pi_n(\cos \theta)]. \quad (15a)$$

For unpolarized incident light, since the scattered energy in the far-field,  $I_s \propto |S_1|^2 + |S_2|^2$ , the normalized scattering phase function can be derived in a form

$$P(\cos \theta) = \frac{|S_1|^2 + |S_2|^2}{\sum_{n=1}^{\infty} (2n+1)(|a_n|^2 + |b_n|^2)}. \quad (16)$$

Another often used parameter in the radiative transfer calculation is the asymmetry factor, which is defined as

$$g = \frac{1}{2} \int_{-1}^1 P(\cos \theta) \cos \theta d \cos \theta. \quad (17)$$

Using Eq. (19), the asymmetry factor can be also expressed in an analytic form

$$g = \frac{2 \sum_{n=1}^{\infty} \left[ \frac{n(n+2)}{n+1} \operatorname{Re}(a_n a_{n+1}^* + b_n b_{n+1}^*) + \frac{2n+1}{n(n+1)} \operatorname{Re}(a_n b_n^*) \right]}{\sum_{n=1}^{\infty} (2n+1)(|a_n|^2 + |b_n|^2)}. \quad (18)$$

## Summary and Conclusions

The light scattering and absorption by a spherical particle in an absorbing medium has been formulated in this study. Using the electromagnetic fields at the particle surface, we have derived the analytic expressions for the single-scattering properties of the particle, which include the absorption, scattering, and extinction efficiencies. In particular, the absorption efficiency is directly derived using the internal field, which leads to a concise analytic formula of  $Q_a$ . Our approach here using the near field avoids the

difficulty to derive the extinction based on the optical theorem using the far field for the case of absorbing media. In this study, the equations for the scattering phase function and asymmetry are based on the far-field approximation.

## Acknowledgments

We wish to thank Dr. P. Chylek for useful discussions. The research work contained herein has been supported by U.S. Department of Energy Grant DE-FG02-97ER62363.

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