Diffusion and Scattering in Multifractal Clouds

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Turbulent cascade processes generically give rise to extremely variable multifractals, and there is now abundant empirical evidence that both cloud liquid water density and cloud radiance fields are multifractal over wide ranges of scale. Since general circulation models and NWPs assume that clouds are on the contrary horizontally homogeneous, it is therefore important to study the transport properties (particle, photon) of multifractals and compare the results with the homogeneous plane parallel situation. Since the plane parallel and multifractal radiative properties diverge for increasingly thick clouds, it becomes fundamentally important to characterize the scaling properties and optical thickness of real clouds.

We investigate these radiative properties of multifracal clouds using two different approaches. In the first, we examine the simplest nontrivial transport problem, diffusion, by considering the scaling properties of one dimensional random walks on media with multifractal diffusivities. We show both theoretically and numerically that the anomalous scaling depends on the scaling of the reciprocal spatially averaged multifractal resistance to diffusion. We show that the anomalous scaling will always be subdiffusive; the walkers are effectively trapped in a hierarchy of barriers. In the small-scale multifractal limit, the trapping is dominated by contributions from a specific order of singularity; this leads to a phase transition between anomalous and normal diffusion as the extreme resitivities (barriers) are reduced.

A second approach considers the scattering statistics associated with radiative transport. We develop an (analogue) co-dimension formalism for dealing with this problem. The basic quantities are the size of the medium measured in units of mean free paths (K), the co-dimension function of the media density, and the scattering (analogue) co-dimension function. We obtain simple relations asymptotic in K between these functions and hence obtain the single scattering statistics. In some special cases, it is possible to renormalize the multifractal so as to obtain an equivalent homogeneous medium. We then estimate anomalous diffusion exponents.

Diffusion

Consider a one-dimensional multifractal optical density field denoted $\rho\lambda(x)$ where $\lambda > 1$ is the ratio of the largest scale of interest to the smallest scale of homogeneity (see Silas 1994 and Lovejoy et al. 1995a for more details). We have

$$\rho_\lambda \sim \lambda^\gamma$$

where the statistical exponents γ are orders of singularity satisfying a well-defined probability distribution at each scale.

$$\Pr(\rho_{\lambda>} > \lambda^\gamma) \approx \lambda^{-c(\gamma)}$$

where $c(\gamma)$ is the co-dimension function and equality is to within slowly varying prefactors. The statistical moments of the multifractal field are described by the moment scaling function $K(q)$

$$\langle \rho_\lambda^q \rangle = \lambda^{K(q)}$$

where q is the order of the moment and the brackets indicate ensemble averaging.
The coefficient of diffusion $D\lambda(x)$ is taken to be

$$D_\lambda(x) = \frac{1}{\rho_\lambda(x)}$$

that is, pλ is a resistance to diffusion; regions of high resistance to diffusion are likely to correspond to rare, dense (impenetrable) regions of the medium. In the diffusion approximation to radiative transfer, pλ(x) is proportional to the (multifractal) optical density (Figures 1 and 2).

We now use a one-dimensional long-time

$$\rho_{eff} = \frac{1}{D_{eff}} = \frac{t}{\overline{x^2}} \approx \frac{1}{N}\sum_{i=1}^{N}\frac{1}{D_i} = \frac{1}{N}\sum_{i=1}^{N}\rho_i$$

where N is the number of distinct sites visited by the random walker and the ρ_i are the resistances associated with those sites, t is the time taken for the walk, and (equation) is the variance of walks on a single realization of the (multifractal) process (overbars indicate means over walks, angle brackets indicate means over ensembles of multifractal realizations of ρ[x]). The simple interpretation is that the random walker experiences an effective resistance $\rho_{eff} = (D_{eff})^{-1}$ equal to the mean resistance of the sites it has visited.
We now define the exponent d_w by

$$\langle\overline{x^2}\rangle \sim t^{\frac{2}{d_w}}$$

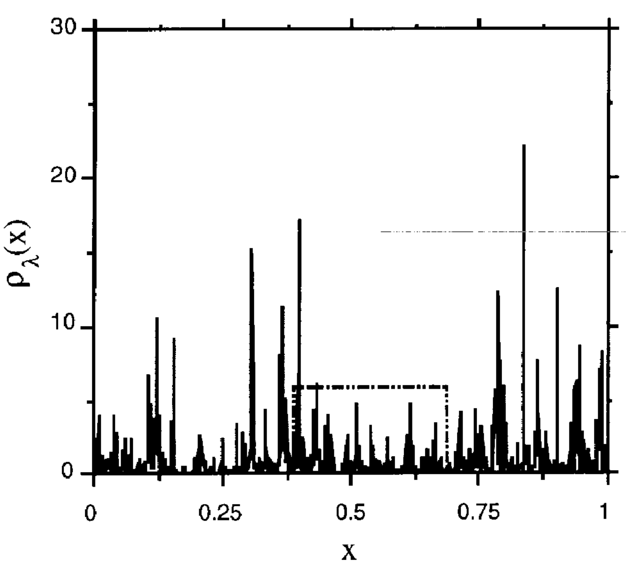

Figure 1. Multifractal field with Cl = 0.2 and λ = 1024.

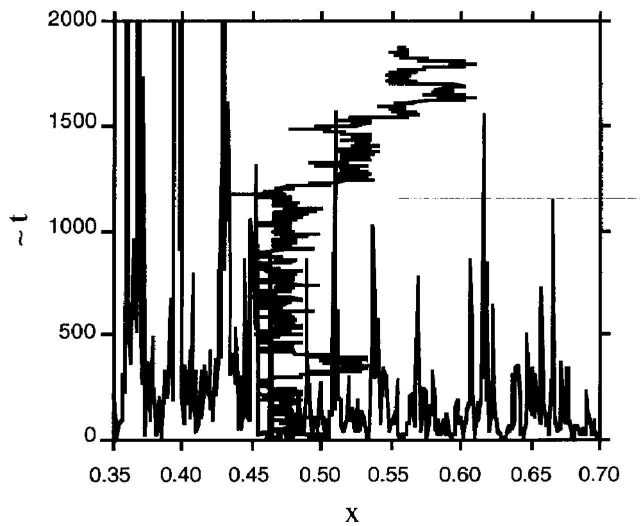

Figure 2. A random walk performed on the inset of Figure 1. A superposition of the trail of the walk (121,080 steps taken) upon the region of the field explored by the walker is pictured here. The walker is "trapped" between large values of the field, hence a slow-down of the diffusion process.

and estimate <Dλ> from the average of the λ intervals:

$$\langle D_\lambda\rangle = \langle \rho_\lambda^{-1}\rangle = \lambda^{K(-1)}$$

where we have assumed that K(-1) is finite (hence will not always apply to the universal multifractals with $\alpha < 2$). We finally obtain

$$d_w = 2 + K(-1)$$

Hence, since $K(-1) > 0$, $d_W > 2$, we will have sub-diffusive behavior; the particles are trapped in a hierarchy of barriers. In the limit $\Lambda\to\infty$ the diffusive behavior is therefore totally dominated by structures with resistivity singularity $\gamma\text{-}1 = K'(-1)$ distributed over a fractal set with co-dimension c(γ-l). The higher order singularities are too rare to affect the transport, and the lower order singularities are too weak to significantly trap the particles. This critical singularity is associated with a phase transition: if the resistivity field is replaced by a thresholded field with all values exceeding a fixed T reset to the value T, then, in the limit $\Lambda\to\infty$, there will be a transition from anomalous diffusion (with the above exponent) to normal diffusion when T is reduced below the critical value $\Lambda^{\gamma-1}$ (Figure 3).

Particle Scatter/Radiative Transfer

Here we outline recent results which provide the basis for systematic study of radiative transport in multifractal media (see Brösamlen 1994 and Lovejoy et al. 1995b). Specifically, we indicate how formulae analogous to the multifractal optical density field arise for radiative properties. Consider the following definitions:

k = extinction coefficient $[m^2Kg^{-1}]$

$<\rho>$ = mean cloud density $[kgm^{-3}]$

l = random photon path distance [m]

L = size of cloud [m]

1_m = mean free path (m.f.p.) of a photon in the equivalent homogeneous cloud = $(k<p>)^{-1}$ [m]

$\lambda = L/1$ = scale ratio $\leq \Lambda$ (Λ = maximum cascade resolution)

$x = l/L = \lambda^{-1}$ random photon nondimensional distance, (fraction of cloud)

$\tau_p = 1/1_m$ = random photon distance, (no. of homogeneous cloud m.f.p.'s = kx)

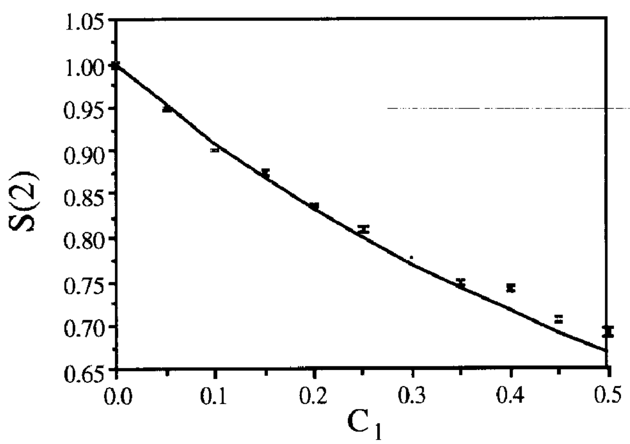

Figure 3. Dependence of the scaling exponent S(2) of the second order moment of x ($<x^2> \sim t^{S(2)}$) on C_l. The solid line is the theory $(1+C_1)^{-1}$; the data points were obtained from simulations.

$K = L/1_m$ = extinction parameter = no. of m.f.p.'s across cloud = mean optical depth = extinction coefficient in units such that $L = 1$, $<p> = 1$.

Figure 4 shows an illustration in one dimension (c.f. Figure 2 for diffusion). The optical distance over a physical distance 1 is thus

$$\tau(l) = \int_l k\rho_\Lambda(z)dz = k\rho_l l$$

where we have written ρl for the average density at resolution 1. We obtain

$$\tau(\gamma,\lambda) = \frac{\rho_\lambda}{\langle\rho\rangle}\lambda^{-1}\frac{L}{l_m} = \lambda^{\gamma-1}\kappa$$

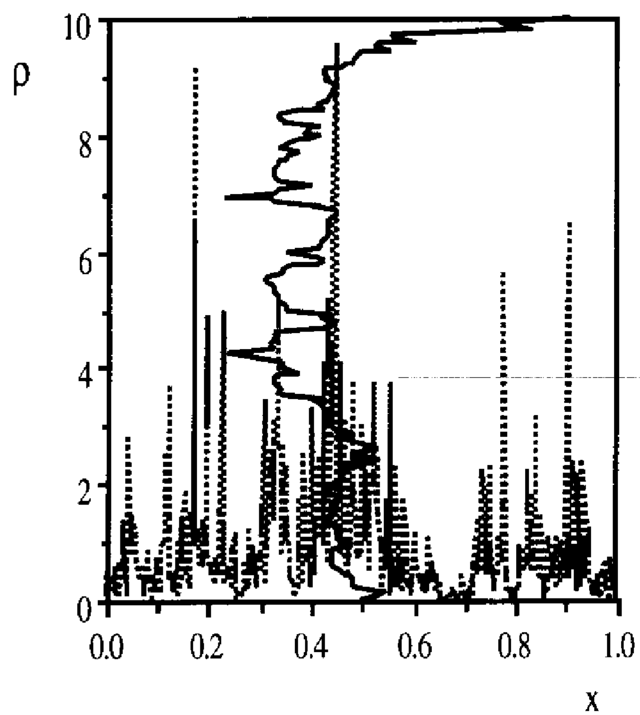

Figure 4. Photon "random walk" in a multifractal cloud (C1= 0.1; A=512) with extinction coefficient K=32. The y-axis represents the cloud density p and the number of scatters of the photon (1 unit corresponds to 20 scatters). The x-axis represents the position in the 1-d cloud. With increasing extinction coefficient K, the mean free path length of the photon decreases.

which is the optical thickness over a distance 1 through a singularity of order γ. The direct (unscattered) transmission T across this distance is thus

$$T(l) = e^{-\tau(l)}$$

Since the transmittance is the probability distribution for photon path lengths, we can average over the singularities and obtain

$$\Pr(1'>l) = \langle T(l) \rangle = \langle e^{-\tau(l)} \rangle$$

Take τp as the dimensionless photon path and write it as a scaling function with an order of singularity τp defined as follows

$$\tau_p = \kappa^{\gamma_p} = \frac{l}{l_m} = \kappa x = \kappa \lambda^{-1}$$

or

$$\lambda = \kappa^{(1-\gamma_p)}$$

since $\lambda > 1$, $K > 1$, we have $\gamma p \leq 1$; also $\Lambda > K^{1-\gamma}P$.

We can now obtain a multifractal scattering formalism in which the extinction coefficient K takes the place of the scaling parameter λ. Instead of the co-dimension function $c(\gamma)$ of the singularities of the cloud density γ, we rather talk about an analogue co-dimension function $cp(\gamma p)$ which describes how the single photon path distance singularity γp varies with the extinction coefficient:

$$\Pr(\tau_p \geq \kappa^{\gamma_p}) = \Pr(1'>l) \approx \kappa^{-c_p(\gamma_p)}$$

$$\langle \tau_p^{q_p} \rangle = f_p(q_p) \kappa^{K_p(q_p)}$$

$f_p(q_p)$ is the prefactor

$$K_p(q_p) = K(q(q_p)) = q_p + q(q_p)$$

$$f_p(q_p) = \frac{q_p \Gamma(-q(q_p))}{1 - K'(q(q_p))} (\log \kappa)^{-1}$$

with q, q_p related as above.

Renormalization

We can now relate the transmission statistics of lognormal multifractal clouds to those of homogeneous clouds. At first sight this seems to be a difficult task since, in the thick limit (K large), both types of clouds will result in a completely different behavior of the radiative transfer properties. However, we find that the photon statistics of a multifractal cloud can be approximated by the photon statistics of a “renormalized” homogeneous cloud in a certain range of photon singularities. This cloud has the direct transmission given by $T(x) = e^{-\kappa_{eff} x}$ where κ_{eff} is the equivalent “effective” extinction coefficient. For a log-normal multifractal cloud, we obtain

$$\kappa_{eff} = \kappa^{\frac{1}{1+c_1}}$$

We now test that this idea works for multiple scattering using the numerical results of Davis et al. (1991) (with C1 = 0.5; Figure 5). These simulations were made using two dimensional discrete lognormal cascades with scale ratio factor 2 per step, total range of scales 2^{10}. Cyclic boundary conditions were used in the horizontal and photons were vertically incident. Using the standard thick cloud plane

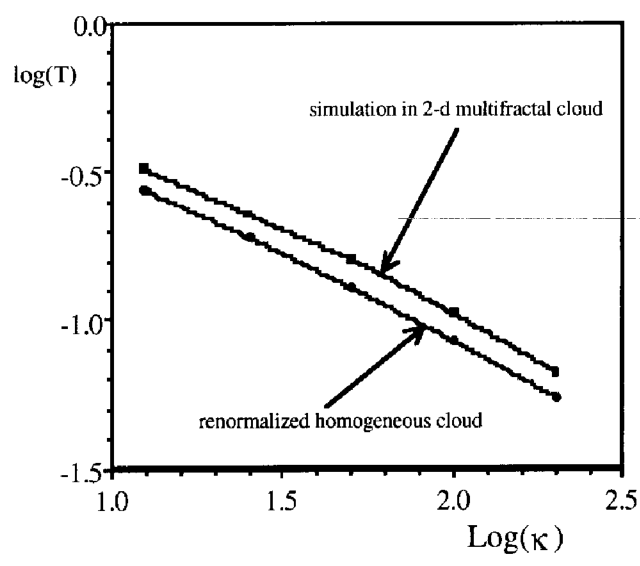

Figure 5. Result on total transmission after multiple scattering through 2-d multifractal cloud (Cl = 0.5) compared with the thick cloud limit of the transmission through a homogeneous cloud with renormalized extinction coefficient K_{eff}.

parallel result with the effective extinction coefficient in place of the true optical thickness $\tau = K_{eff}\langle p \rangle = K_{eff}$, we obtain

$$\langle T \rangle = \frac{1}{1 + \frac{1}{2}\kappa^{\frac{1}{1 + C_1}}} \approx 2\kappa^{-\frac{1}{1 + C_1}}$$

Figure 5 shows the result of superposing this function on the numerics, which are nearly power law even for K as low as 12.5. The total transmittances through the renormalized homogeneous cloud show for all values of K only less than 20% difference from the total transmittances through the multifractal cloud.

Discussion

The surprisingly accurate prediction of thick cloud numerics can perhaps best be understood by considering the relation between radiative transfer and diffusion on multifractals. In general, there will be two significant limits: the large Λ (wide cascade range) and large K (thick cloud) limits. Clearly, for fixed and finite Λ, if the cloud is made thick enough ($K \gg \Lambda$), the mean free path will be much smaller than a single resolution element, and the photons will diffuse through each homogeneous region of size Λ^{-1}. The overall result will be photons diffusing through the multifractal cloud. In actual fact, diffusion can still occur under somewhat less stringent conditions when K is large, the main requirement being that weak density regions become so rare that direct photon transmittance across a large fraction of the cloud is statistically negligible. The multifractals with $\alpha<2$ have precisely the property that they are dominated by weak events (negative singularities) called "Levy holes." It is a priori possible that even with large K–, if Λ is sufficiently large (the order of the limits $\Lambda\to\infty$ and K<°$\to\infty$ is important, i.e., with K fixed, but with $\Lambda\to\infty$), they will have large regions dominated by the holes and hence lead to nondiffusive transfer.

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